L-functions and applications
6. Exercise Sheet

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Groupwork

Exercise G1

Let $f \in \mathbb{Z}[X]$. For a prime p, let $n(p)$ be the number of distinct zeros of f mod p in $\mathbb{F}_p$. Prove that for the right definition of average, the average of $n(p)$ is equal to the number of distinct monic irreducible factors of f in $\mathbb{Q}[X]$.

Hint: You may use the orbit-counting theorem: If G is a finite group that acts on a finite set X and $\#X/G$ denotes the number of orbits of X under this action, then

$$\#X/G = \frac{1}{\#G}\sum_{g\in G}\#\{x \in X : gx = x\}.$$

Exercise G2

(a) Let ρ_1 and ρ_2 be two representations of a Galois group of a Galois extension L/K. Use Chebotarev's density theorem to show that if $\mathscr{L}(L/K,\rho_1,s) = \mathscr{L}(L/K,\rho_2,s)$, then $\rho_1 \cong \rho_2$.

(b) Show that if K and L are finite Galois extensions of $\mathbb{Q}$ and $\zeta_K(s) = \zeta_L(s)$, then $K \cong L$.

Hint: For the second point you should use Theorem 7.5 from the lectures.

Exercise G3

Prove Artin's conjecture on the holomorphicity of $\mathscr{L}(L/K,\rho,s)$ for *any* irreducible representation ρ of degree greater than 1: For this you can look at any S_3 extension $K/\mathbb{Q}$ and the Artin L-function associated to the standard representation St of S_3. Show that $\mathrm{St} = \mathrm{Ind}_{C_3}^{S_3}\theta$ for any non-trivial character θ of C_3. Use this to show that $\mathscr{L}(K/\mathbb{Q},\mathrm{St},s)$ extends to an entire function.