# L-functions and applications
# 7. Exercise Sheet

TECHNISCHE
UNIVERSITÄT
DARMSTADT

**Department of Mathematics**
**Dr. Jolanta Marzec, Dr. Michael Neururer**

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## Groupwork

**Exercise G1**
Let $f = \sum_{n=0}^{\infty} a(n)q^n \in \mathscr{M}_k(\Gamma_0(N), \chi)$ and $L(f,s) = \sum \frac{a(n)}{n^s}$ its L-series.

(a) Using the fact that $g = f|W_N = \sum_{n=0}^{\infty} b(n)q^n \in \mathscr{M}_k(\Gamma_0(N), \overline{\chi})$, where $W = \left(\begin{smallmatrix} & -1 \\ N & \end{smallmatrix}\right)$, show that $L(f,s)$ has meromorphic continuation to $\mathbb{C}$ and $\Lambda(f,s) := N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$ satisfies

$$\Lambda(f,s) = i^k \Lambda(g, k-s)$$

with (possibly) simple poles at $s = 0$ and $s = k$ of residues $a_0$ and $-b_0$ respectively.

(b) Assuming that $f$ is a newform and using the fact that $a(n) = O(n^{k/2})$, show that for $\mathrm{Re}\, s > \frac{k}{2} + 1$

$$L(f,s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - a(p)p^{-s} + \chi(p)p^{k-1-2s}}.$$

**Exercise G2** (Eisenstein series)
Let $k \geq 1$ and $\phi$ and $\psi$ be two Dirichlet characters modulo $N_1$ and $N_2$ respectively with $\phi(-1)\psi(-1) = (-1)^k$. The Eisenstein series

$$E_k^{\phi,\psi}(\tau) = c(k,\phi,\psi) + \sum_{n \geq 1} \left( \sum_{m|n} \phi(n/m)\psi(m)m^{k-1} \right) e^{2\pi i n \tau},$$

where $c(k,\phi,\psi) \in \mathbb{C}$ is an explicit constant that is not important for this exercise, defines a modular form in $\mathscr{M}_k(\Gamma_0(N_1N_2), \phi\psi)$ (you can use this as a fact). Show that

$$L(E_k^{\phi,\psi}, s) = L(\phi, s)L(\psi, s-k+1)$$

and hence in particular, if $\phi(-1)\psi(-1) = -1$, $L(E_1^{\phi,\psi}) = \mathscr{L}(\mathbb{Q}(\zeta_{N_1N_2})/\mathbb{Q}, \phi \oplus \psi, s)$. Here we interpret $\phi$ and $\psi$ as characters of $\mathrm{Gal}(\mathbb{Q}(\zeta_{N_1N_2})) = (\mathbb{Z}/N_1N_2\mathbb{Z})^\times$ using the fact that $\mathrm{Gal}(\mathbb{Q}(\zeta_{N_1N_2}))$ surjects both onto $(\mathbb{Z}/N_1\mathbb{Z})^\times$ and $(\mathbb{Z}/N_2\mathbb{Z})^\times$. This gives the first, and easiest, example of Theorem 9.5 from the lectures.

**Exercise G3** (☠)
Let $\rho$ be an even irreducible 2-dimensional representation of a finite Galois group $\mathrm{Gal}(L/\mathbb{Q})$. Let $K_0(y) = \int_0^\infty \cos(y \sinh t)dt$ be a $K$-Bessel function. Let $a_n$ be the Dirichlet coefficients of $\mathscr{L}(L/\mathbb{Q}, \rho, s)$ and define

$$F(x+iy) = \sum_{n \geq 1} \frac{a_n}{\sqrt{n}} y^{1/2} K_0(2\pi n y)\cos(2\pi n x)$$

as a function on the upper half plane. Show that there exists $N \geq 1$ and a Dirichlet character modulo $N$ such that $F$ is modular of level $N$ in the following sense:

$$\forall_{\left(\begin{smallmatrix} a & b \\ c & b \end{smallmatrix}\right) \in \Gamma_0(N)} F\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)F(\tau).$$

The function $F$ is not holomorphic but it is an eigenfunction of the differential operator $-y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ with eigenvalue 1/4. Such functions are called **Maass forms** and this exercise is wide open.