L-functions and applications
4. Exercise Sheet

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Groupwork

Exercise G1 (Prime splitting in towers of extensions)

(a) Let $K \subset L \subset M$ be extensions of number fields and $\mathfrak{p}$ a prime in $\mathcal{O}_K$. If $\mathfrak{P} \triangleleft \mathcal{O}_L$ lies above $\mathfrak{p}$ and $\mathfrak{s} \triangleleft \mathcal{O}_M$ lies above $\mathfrak{q}$

$$e_{\mathfrak{s}/\mathfrak{p}} = e_{\mathfrak{s}/\mathfrak{q}} e_{\mathfrak{q}/\mathfrak{p}} \quad \text{and} \quad f_{\mathfrak{s}/\mathfrak{p}} = f_{\mathfrak{s}/\mathfrak{q}} f_{\mathfrak{q}/\mathfrak{p}}$$

(b) Describe how 7 decomposes in $K = \mathbb{Q}(\sqrt{5}, \sqrt{7}, \sqrt{11})$, i.e., find the ramification, inertia degree and the number of primes above 7 in K.

Exercise G2 (Prime splitting in cyclotomic fields)
The aim of this exercise is to prove the following proposition which describes splitting behaviour of primes in **cyclotomic fields**, i.e., in $\mathbb{Q}(\mu_m)$, where μ_m is a primitive m-th root of unity. The consecutive steps of the proof may be illustrated with use of SageMath.

Proposition *Let $m = \prod_p p^{\nu_p}$ be the prime factorization of m and, for every $p \in \mathbb{P}$, let f_p be the smallest positive integer such that $p^{f_p} \equiv 1 \bmod \frac{n}{p^{\nu_p}}$. Then one has in $K = \mathbb{Q}(\mu_m)$ the factorization*

$$p = (\mathfrak{p}_1 \cdots \mathfrak{p}_r)^{\phi(p^{\nu_p})},$$

where $\mathfrak{p}_1 \dots \mathfrak{p}_r$ are distinct prime ideals, all of degree f_p, and ϕ denotes the Euler function.

(a) Let $K = \mathbb{Q}(\mu)$, μ a primitive m-th root of unity. Assume that m is a power of some prime number p. Use a minimal polynomial of μ and its decomposition into irreducible factors over K to prove that

$$p\mathcal{O}_K = (1-\mu)^{\phi(m)}\mathcal{O}_K .$$

Use the fundamental equality (see cheatsheet II) to deduce that

$$\mathcal{O}_K/(1-\mu)\mathcal{O}_K \cong \mathbb{Z}/p\mathbb{Z} .$$

SAGE: Consider cyclotomic fields for $m = 4, 27, 5$. Find their degree over $\mathbb{Q}$ and defining polynomials; factor these polynomials over $\mathbb{Q}(\mu)$; find decomposition of $p = 2, 3, 5$ in corresponding cyclotomic fields. It may be helpful to use the following code and the code from exercise G4:

```
m =
K.<zetam> = CyclotomicField(m) #zetam is a primitive m-th root of unity
zetam.minpoly() #minimal polynomial of zetam
```

(b) Under assumptions from (a), compute the discriminant of the basis $1, \mu, \dots, \mu^{\phi(m)-1}$ of $K/\mathbb{Q}$.
SAGE: Compute the discriminant of the ideal generated by the above basis for K as in (a). Use the functions `absolute_norm()` and `discriminant()`.

(c) Deduce from (a) that $\mathcal{O}_K = \mathbb{Z}[\mu] + (1-\mu)^t\mathcal{O}_K$ for any $t \geq 1$. Use this and the fact that $\operatorname{disc}(1, \mu, \dots, \mu^{\phi(m)-1})\mathcal{O}_K \subseteq \mathbb{Z}[\mu]$ to prove that $O_K = \mathbb{Z}[\mu]$.
SAGE: Verify the statement $O_K = \mathbb{Z}[\mu]$ for K as in (a).

(d) Now let m be an arbitrary positive integer. Prove that $O_K = \mathbb{Z}[\mu]$.
SAGE: Verify this statement for $K = \mathbb{Q}(\mu_m)$ with $m = 4 \cdot 9 \cdot 5$.

(e) Use Kummer-Dedekind theorem (see cheatsheet II) to prove the above Proposition.
SAGE: Factor first twenty prime numbers in $\mathcal{O}_K$ for K as in (d) to verify this statement.

Hints:
(b): Let Φ_m be the minimal polynomial of μ. Relate $\mathrm{disc}(1, \mu, \ldots, \mu^{\phi(m)-1})$ to $\Phi_m'(\mu)$. Then differentiate the equation $(X^{m/p} - 1)\Phi_m(X) = X^m - 1$ at $X = \mu$.
(d): You may use the following basic fact: Let L and L' be two Galois extensions of $\mathbb{Q}$ of degree n, resp. n', such that $L \cap L' = \mathbb{Q}$. Let $\omega_1, \ldots, \omega_n$, resp. $\omega_1', \ldots, \omega_n'$ be an integral basis of $L/\mathbb{Q}$, resp. $L'/\mathbb{Q}$, with discriminant d, resp. d'. If d and d' are coprime, then $\{\omega_i \omega_j' : i = 1, \ldots, n; j = 1, \ldots, n'\}$ is an integral basis of LL', of discriminant $d^{n'} d'^{n}$.
(e): First show that for $m = p^\nu M$, $p \nmid M$, the minimal polynomial $\Phi_m(X)$ of primitive m-th root of unity satisfies $\Phi_m(X) \equiv \Phi_M(X)^{\phi(p^\nu)}$ mod p. Then reduce to the case $p \nmid m$ and show that $\Phi_m(X)$ doesn't have multiple roots mod p as a divisor of $X^m - 1$. Observe that $\mathbb{F}_{p^{f_p}}$ is the splitting field of the polynomial Φ_m mod p.

Exercise G3
Let L/K be a Galois extension such that $\mathrm{Gal}(L/K)$ is not cyclic. Show that no prime $\mathfrak{p}$ of K is inert, i.e. $\mathfrak{p}\mathcal{O}_L$ is never a prime ideal in L.

Hint: Show that if p is inert, the decomposition group of p, $\mathrm{Gal}(L/K)$, is isomorphic to the Galois group of a finite field extension. These are all cyclic.

Exercise G4 (A cubic extension)
Let $L = \mathbb{Q}(\alpha)$ be the cubic extension of $\mathbb{Q}$ generated by α with minimal polynomial $P(X) = X^3 - X - 1$. In this exercise you should use SageMath to help with the calculations. Here is a code snipped to get you started.

```
Pol.<x> = PolynomialRing(ZZ) #Defines a polynomial ring
P = x^3-x-1
L.<a> = NumberField(P) #Defines the number field generated by a root of P
OL = L.ring_of_integers()
OL.basis()
L.disc()
L.factor(5) #Gives the prime factorisation of (5) in L
P.factor_mod(5) #Factors P modulo 5. This is closely connected to L.factor(5).
```

(a) Let H be the splitting field of P and let $\alpha_1, \alpha_2, \alpha_3$ be the roots of P in $\mathbb{C}$. Show that $\sqrt{\mathrm{disc}(L)} \in H$ and conclude that L is not a Galois extension of $\mathbb{Q}$.

(b) Show that a prime p can split in L in the following ways:

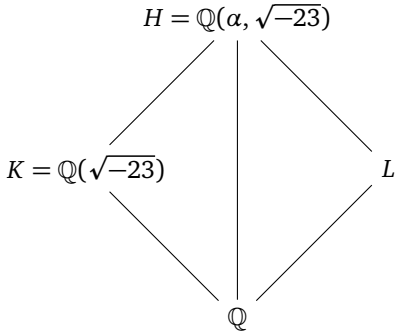
$$p\mathcal{O}_L = \begin{cases} \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3 & \text{with } e_{\mathfrak{p}_i/p} = f_{\mathfrak{p}_i/p} = 1, \\ \mathfrak{p}_1\mathfrak{p}_2 & \text{with } e_{\mathfrak{p}_i/p} = f_{\mathfrak{p}_1/p} = 1,\ f_{\mathfrak{p}_2/p} = 2, \\ \mathfrak{p} & \text{with } e_{\mathfrak{p}/p} = 1,\ f_{\mathfrak{p}/p} = 3 \\ \mathfrak{p}_1\mathfrak{p}_2^2, & \text{if } p = 23, \text{with } e_{\mathfrak{p}_1/p} = f_{\mathfrak{p}_i/p} = 1, e_{\mathfrak{p}_2/p} = 2. \end{cases}$$

Find examples for all of these cases. Consider the Euler product

$$\zeta_L(s) = \prod_p \zeta_{L,p}(s),$$

and write down what $\zeta_{L,p}(s)$ is, depending on the splitting behaviour of p. Show that $L(s) = \zeta_L(s)\zeta(s)^{-1}$ is meromorphic on $\mathbb{C}$ and holomorphic in $\mathrm{Re}\, s \geq 1$ and write down an Euler product of this L-function.

(c) Consider the following diagram of extensions:



Which of the extensions you see are Galois? Write down the corresponding Galois groups.

(d) Show that p is inert in K, i.e., $p\mathcal{O}_K$ is a prime ideal, if and only if $p\mathcal{O}_L = \mathfrak{p}_1\mathfrak{p}_2$.

(e) Show that

$$\zeta_H(s) = L(s)^2\zeta_K(s),$$

by comparing local Euler factors.

☠ Show that $L(s)$ extends to a holomorphic function on $\mathbb{C}$ or, in other words, every zero of $\zeta(s)$ is also a zero of $\zeta_L(s)$.