L-functions and applications
2. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics **SS 2019**
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Groupwork

Exercise G1 (Special values of ζ)
The Bernoulli numbers B_n can be defined as the Taylor coefficients of

$$f(y) = \frac{1}{e^y - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} y^{n-1}$$

(a) Use the geometric series expansion $f(y) = \sum_{n=1}^{\infty} e^{-ny}$ to show $\mathscr{M}(f,s) = \Gamma(s)\zeta(s)$.

(b) Use Theorem 2.5 to deduce that $\zeta(s)$ has a holomorphic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole of residue 1 at $s = 1$. Use the same theorem to determine $\zeta(-k)$ for $k \in \mathbb{Z}_{\geq 0}$ and show that $\zeta(-2k) = 0$ for all $k \geq 1$. These are called the *trivial zeros* of the zeta function.

(c) Use the functional equation to determine $\zeta(2k)$ for $k \geq 1$.

Comment: In several popular mathematics sources the fact that $\zeta(-1) = -1/12$ is often (erroneously) stated as "$1 + 2 + 3 + \ldots = -1/12$".

Exercise G2 (Zeros of ζ)
Show the following statements:

- The non-trivial zeros of ζ must lie in the critical strip $0 < \mathrm{Re}\, s < 1$.
- Show that for every non-trivial zero s_0 of ζ also $1 - s_0$, $\overline{s_0}$ and $1 - \overline{s_0}$ are zeros.
- ☠ Finish Riemann's work and show that in fact all non-trivial zeros of ζ lie on the line $\mathrm{Re}\, s = 1/2$. This is known as the *Riemann hypothesis*.

> ...Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.
>
> Riemann 1859, in "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"

Exercise G3
Let $\zeta_6 = e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{-3}}{2}$ and $K = \mathbb{Q}(\sqrt{-3})$.

(a) Provide a formula for trace and norm of elements in K.

(b) Prove that $\mathscr{O}_K = \mathbb{Z}[\zeta_6]$.

(c) Compute the discriminant of K.

(d) Find all units in $\mathbb{Z}[\zeta_6]$. Write $\mathscr{O}_K^\times$ as a product of a cyclic group and a certain abelian group.

(e) Find an integral basis of an ideal $(\sqrt{-3})$ and compute its discriminant and the absolute norm. Do the same for the ideal (3).

(f) Compute the inverse different of $K/\mathbb{Q}$ and its inverse (the different).

(g) Verify your answers in SageMath. Once you define the number field K (see short explanation in the sheet `2 Regular primes.ipynb`), you can use the following functions implemented in SageMath: `K.discriminant()`, `K.unit_group()`, `1/(K.different())`; in order to define an ideal (3), type `I=K.ideal(3)`, and to find its integral basis `I.basis()`. More functions can be found here: `http://doc.sagemath.org/html/en/reference/number_fields/index.html` .

Hint: For b. use Gauss's lemma: A non-constant polynomial $f \in \mathbb{Z}[x]$ is irreducible if and only if it is irreducible in $\mathbb{Q}[x]$ and the greatest common divisor of the coefficients of f is 1.