L-functions and applications
3. Exercise Sheet

TECHNISCHE UNIVERSITÄT DARMSTADT

Department of Mathematics **SS 2019**
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Groupwork

Exercise G1 (Lecture: Proposition 3.25)
Let K be a number field and

$$\zeta_K(s) = \sum_{0 \neq \mathfrak{a} \triangleleft \mathcal{O}_K} \frac{1}{\mathfrak{N}(\mathfrak{a})^s},$$

its Dedekind zeta function. Prove that the series $\zeta_K(s)$

(a) converges absolutely and uniformly in compact subsets of the domain $\mathrm{Re}(s) > 1$.

(b) has an Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - \mathfrak{N}(\mathfrak{p})^{-s}},$$

where $\mathfrak{p}$ runs through the prime ideals of K.

The Kronecker symbol: A fundamental discriminant is the discriminant of a quadratic field, i.e., either $D \equiv 1 \bmod 4$ and D is square-free or $D \equiv 2, 3 \bmod 4$ and $D/4$ is square-free. For a fundamental discriminant D we define the Kronecker character $\chi_D : \mathbb{N} \to \{-1, 0, 1\}$ in two steps. First, for an odd prime p, we define the Legendre symbol

$$\left(\frac{d}{p}\right) := \begin{cases} 0 & \text{if } p \mid d, \\ 1 & \text{if } d \text{ is a square modulo } p, \\ -1 & \text{if } d \text{ is not a square modulo } p. \end{cases}$$

The character χ_D is defined to be a completely multiplicative function which at prime numbers is given is given by the following formulae:

$$\chi_D(2) := \begin{cases} 0 & \text{if } D \equiv 0 \bmod 4, \\ 1 & \text{if } D \equiv 1 \bmod 8, \\ -1 & \text{if } D \equiv 5 \bmod 8. \end{cases}$$

$$\chi_D(p) := \left(\frac{D}{p}\right) \quad \text{if } p \text{ is an odd prime.}$$

Exercise G2 (Prime ideals in quadratic fields)
Let $K = \mathbb{Q}(\sqrt{m})$ with m square-free (but possibly negative) be a quadratic field and

$$\mathcal{O}_K = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}\sqrt{m} & \text{if } m \equiv 2, 3 \bmod 4, \\ \mathbb{Z} \oplus \mathbb{Z}\frac{1+\sqrt{m}}{2} & \text{if } m \equiv 1 \bmod 4. \end{cases}$$

its ring of integers.

(a) Check that $D = \mathrm{disc}(K) = 4m$ if $m \equiv 2, 3 \bmod 4$ and $D = m$ if $m \equiv 1 \bmod 4$.

(b) Show that

$$p \begin{cases} \text{is inert} \\ \text{ramifies} \\ \text{splits(completely)} \end{cases} \text{in } K \quad \Longleftrightarrow \quad \chi_D(p) = \begin{cases} -1, \\ 0, \\ 1. \end{cases}$$

(c) Now let $K = \mathbb{Q}(i)$. You may assume that $\mathcal{O}_K = \mathbb{Z}[i]$ is a principal ideal domain (in fact it is even a euclidean domain). Show that the above statements on prime decompositions imply the following fact: If a prime p is congruent to 1 modulo 4 it can be written as a sum of two squares and if a prime is congruent to 3 modulo 4 it cannot be written as a sum of two squares.

Exercise G3 (Dedekind zeta functions of quadratic fields)
Let $K = \mathbb{Q}(\sqrt{m})$. Show that for $\operatorname{Re} s > 1$,

$$\zeta_K(s) = \zeta(s) \prod_p (1 - \chi_D(p) p^{-s})^{-1} = \zeta(s) L(\chi_D, s).$$

Use this to show that the number of times that a number n is the norm of an ideal in $\mathcal{O}_K$ is given by

$$\sum_{m \mid n} \chi_D(m).$$

Exercise G4 (Dedekind zeta function of $\mathbb{Q}(i)$)

(a) Show that for $\operatorname{Re} s > 1$,

$$\zeta_{\mathbb{Q}(i)}(s) = \frac{1}{4} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{0\}} \frac{1}{(m^2 + n^2)^s}.$$

(b) Let $r_2(n)$ be the number of times n can be written as a sum of two integer squares. Show that

$$r_2(n) = 4 \sum_{m \mid n} \chi_{-4}(m).$$

(c) Use Leibniz formula for π to see that $L(\chi_{-4}, 1) = \frac{\pi}{4}$. Show that when $s \to 1^+$,

$$\log L(\chi_{-4}, s) = \sum_{p \equiv 1 \bmod 4} p^{-s} - \sum_{p \equiv 3 \bmod 4} p^{-s} + O(1),$$
$$\log(1 - 2^{-s}) \zeta(s) = \sum_{p \text{ odd}} p^{-s} + O(1).$$

(d) Combine the above two formulae to deduce that both

$$\sum_{p \equiv 1 \bmod 4} p^{-s} \quad \text{and} \quad \sum_{p \equiv 3 \bmod 4} p^{-s}$$

tend to ∞ as $s \to 1^+$. This gives a new proof of the fact that there are infinitely many primes in each of $1 + 4\mathbb{Z}$ and $3 + 4\mathbb{Z}$.

Let $\pi_{1,4}(x)$ and $\pi_{3,4}(x)$ denote the primes $< x$ that are congruent to 1 and 3 modulo 4 respectively. If you have not done so already, check with SageMath that usually $\pi_{3,4}(x)$ is greater than $\pi_{1,4}(x)$. Quantify this statement by showing

$$\frac{1}{\log X} \sum_{\substack{n < X \\ \pi_{3,4}(n) > \pi_{1,4}(n)}} \frac{1}{n} \xrightarrow{X \to \infty} 0.9959\ldots$$

This is called Chebyshev's bias and was shown by Rubinstein and Sarnak in 1994 only under the assumption of the Generalised Riemann Hypothesis and another conjecture on the zeros of the Riemann zeta function.