# L-functions and applications
# 5. Exercise Sheet

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## Groupwork

**Exercise G1**

(a) Find a polynomial $f \in \mathbb{Z}[X]$ with no root in $\mathbb{Z}$ but with a root modulo every prime.

(b) Show that $X^4 + 1$ is irreducible in $\mathbb{Z}[X]$ but reducible modulo every prime. **Hint:** Try to use G3 of exercise sheet 4.

**Exercise G2** (Infinitely many primes split completely)
In this exercise we show that in any non-trivial extension $K/\mathbb{Q}$ infinitely many primes split completely.

(a) First reduce the statement to the case where $K/\mathbb{Q}$ is a Galois extension.

(b) Let $K = \mathbb{Q}(\alpha)$, with an algebraic integer $\alpha$ and $f \in \mathbb{Z}[X]$ be the minimal polynomial of $\alpha$. Use the Kummer–Dedekind Theorem to show that for almost all primes $p$ we have the following equivalence

$$p \text{ splits completely} \Longleftrightarrow f \bmod p \text{ has a root.}$$

(c) $f$ has a root modulo $p$ if there exists $n \in \mathbb{Z}$ such that $p | f(n)$. Suppose the set $P$ of such $p$ is finite and enlarge $P$ to contain all primes dividing $\operatorname{disc}(f)$. Let $t$ be a positive integer such that $\operatorname{ord}_p(t) > \operatorname{ord}_p(f(0))$ for all $p \in P$. Show that $\operatorname{ord}_p f(mt) = \operatorname{ord}_p(f(0))$ for all $p \in P$ and all $m \in \mathbb{Z}$ and derive a contradiction to the assumption that $P$ is finite.

(d) Consider $K = \mathbb{Q}(\zeta_m)$. Show that the above results imply a special case of Dirichlet's theorem: there are infinitely many primes that are congruent to 1 modulo $m$.

**Exercise G3** (Corollary 3.47 in lecture notes)
Let $K$ be a number field and $\alpha \in \mathcal{O}_K$ such that $\alpha \bmod \mathfrak{p}$ is a square in $\mathcal{O}_K/\mathfrak{p}$ for almost all prime ideals $\mathfrak{p}$. Then $\alpha$ is a square.

**Exercise G4** (A curious $L$-function: Continued)

(a) Compute the first 1000 coefficients of the $L$-series $L$ from exercise G4 of exercise sheet 4.

(b) Compute the first 1000 coefficients of the power series

$$f(q) = q \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{23n})$$

and compare them to the Dirichlet coefficients of $L$.

☠ Prove that for a prime $p \neq 23$, the $p$-th Dirichlet coefficient $a_p$ of $L$ is equal to the $p$-th coefficient of the power series $f(q)$.
This is a special case of Serre's modularity conjecture, which was proved in 2008 by Khare–Wintenberger.

**Exercise G5** (Regular representation)
Let $G$ be a finite group with identity element $1_G$.

(a) Compute the character $\chi$ of the (left) regular representation on $\mathbb{C}[G]$.

(b) Let $\rho$ be an irreducible representation of $G$ with character $\chi_\rho$. Compute the hermitian inner product $(\chi, \chi_\rho)$ and deduce that

$$\dim_{\mathbb{C}} \mathbb{C}[G] = \sum_{(\rho,V)} (\dim_{\mathbb{C}} V)^2,$$

where the sum runs over irreducible representations of $G$.

(c) Prove that $\mathrm{Ind}_{\{1_G\}}^G \mathbf{1} = \mathbb{C}[G]$.

**Exercise G6**

Let $G = \{1, g\}$ be the cyclic group of order 2.

(a) Prove that $\rho : h \mapsto \left(\begin{smallmatrix} 3 & -4 \\ 2 & -3 \end{smallmatrix}\right)$ defines a 2-dimensional representation of $G$.

(b) Find two 1-dimensional subrepresentations $\sigma_1$, $\sigma_2$ of $\rho$ such that $\rho = \sigma_1 \oplus \sigma_2$.

**Hint**: Find eigenvectors $v_1$, $v_2$ of $\rho(g)$.